Approach to equilibrium in *N*-body gravitational systems

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(Received 21 January 1998)

The evolution of closed gravitational systems is studied by means of *N*-body simulations. This, as well as being interesting in its own right, provides insight into the dynamical and statistical mechanical properties of gravitational systems: the possibility of the existence of stable equilibrium states and the associated relaxation time would provide an ideal situation where relaxation theory can be tested. Indeed, these states are found to exist for *single* mass *N*-body systems, and the condition for this is simply that obtained from elementary thermodynamical considerations applied to self-gravitating ideal gas spheres. However, even when this condition is satisfied, some initial states may not end as isothermal spheres. It is therefore only a necessary condition. Simple considerations also predict that, for fixed total mass, energy, and radius, stable isothermal spheres are unique. Therefore, statistically irreversible perturbations to the density profile, caused by the accumulation of massive particles near the center of multimass systems, destroy these equilibria if the aforementioned quantities are kept fixed. The time scale for this to happen was found to be remarkably short (a few dynamical times when $N=2500$) in systems undergoing violent relaxation. The time taken to achieve thermal equilibrium depended on the initial conditions and could be comparable to a dynamical time (even when the conditions for violent relaxation were not satisfied) or the two body relaxation time. The relaxation time for velocity anisotropies was intermediate between these two time scales, being long compared to the dynamical time but much (about four times) shorter than the time scale of energy relaxation. This last result, along with the observation of the anomalously rapid mass segregation in some situations, suggests that, in gravitational systems, different quantities may relax at different rates, and that the thermal (two body) relaxation time scale, even if accurate for energy relaxation of single mass systems, may not be universal. This in turn indicates that the issue of relaxation in gravitational systems is far from being a closed subject. [S1063-651X(98)10209-X]

PACS number(s): 05.70.Ln, 98.10.+z, 02.60.−x, 03.20.+i

I. INTRODUCTION AND MOTIVATION

The peculiarity of gravitational interactions compared to those governing laboratory systems can be illustrated by the following example. An orbiting satellite on a circular path loses energy (by dissipation say) and drops to another (also circular) orbit with a smaller radius. The result is that its circular velocity *increases*, even though its total energy decreased. This situation is easily translated into the language of large *N*-body systems in virial equilibrium. In this case, the kinetic and potential energies are related by $2T+V=0$, which implies that the total energy $E=-T$. Decreasing the total energy will therefore be equivalent to increasing the kinetic energy [1–3]. That is, the system has *negative specific heat*—since taking energy away from it heats it up. If one is able to apply this logic to subsystems of a gravitational configuration in dynamical equilibrium, one can see that ''heat'' will *effectively* flow from hotter regions (those having larger average random kinetic energy) to cooler ones—which implies that any temperature gradient is enhanced instead of being erased.

Any increase in the kinetic energy of a system in virial equilibrium is bound (by the virial relation) to decrease its potential energy. In a spherical system of mass M and radius R, the potential energy is given by

$$V=-\frac{\gamma GM^2}{R}, \tag{1}$$

where γ is a constant which depends on the density distribution of the system (and is larger for centrally concentrated objects). Decreasing the potential energy will therefore imply a decrease in radius or an increase in central concentration. If a situation arises where a gravitational system can, to a first approximation, be considered to be composed of a centrally concentrated core in virial equilibrium which is hotter than a surrounding shell, the core will lose energy to the surrounding shell and contract, while getting still hotter. If the surrounding halo cannot heat fast enough, we have a runaway instability, with the system evolving toward states that are less and less homogeneous in both physical and velocity spaces [1,2]. This phenomenon has been termed a ''gravothermal catastrophe'' by Lynden-Bell and Wood [4]. A system undergoing such evolution, instead of tending toward a most probable final equilibrium state, can increase its Boltzmann entropy indefinitely—simply by evolving a denser core and a more diffuse halo [1].

Thermal equilibrium configurations do nevertheless exist for gravitational systems. For open (spatially unbound) systems, however, these turn out to be infinite. In a closed spherical system, the boundary has the effect of adding an external pressure term to the virial equation. The kinetic and potential energies are then related by $2T/V\geqslant 1$. If this ratio is very large, the behavior will be similar to that of a normal laboratory system—and a constant temperature (almost) spatially homogeneous equilibrium configuration will exist. If, on the other hand, the potential energy is comparable to the kinetic energy, then peculiar effects related to the nature of

gravitational interactions will dominate. It turns out that the precise criterion for avoiding gravothermal instability in an *ideal gas* sphere is [4–6]

$$\mu = \frac{ER}{GM^2} \geqslant -0.335. \tag{2}$$

Enclosed gas spheres for which Eq. (2) holds may reach constant temperature equilibria with a well defined asymptotic statistical distribution. These are often referred to as (Lane-Embden) isothermal spheres, because their final distribution functions obey the usual Maxwell-Boltzmann statistics of an isothermal gas. It is easy to see that for systems in virial equilibrium $ER/GM^2 = -\gamma/2$. In the case when the density decreases with radius, γ cannot be smaller than $\frac{3}{5}$. Almost all systems in virial equilibrium therefore do not tend toward thermal equilibrium states—since dynamical equilibria of self-gravitating are necessarily strongly inhomogeneous, with density decreasing with radius. In such systems the central core may behave almost as if it is an independent subsystem in virial equilibrium, and its (rather weak but important) interaction with the outer regions will lead to a gravothermal catastrophe.

Maximum entropy solutions of the collisionless Boltzmann equation (CBE) which have constant velocity dispersions also obey Maxwell-Boltzmann statistics and, by analogy, are also referred to as isothermal spheres. Intuitively, therefore, one would expect condition (2) to hold for spherical stellar systems [1]. Indeed, a *local* stability analysis based on a detailed treatment of the statistical mechanics of gravitating systems seems to confirm this [7,8]. However, the aforementioned solutions of the CBE are obtained by constraining the energy, not the temperature. Therefore, strictly speaking, they represent constant-energy solutions, rather than isothermal solutions in the standard sense.

By global minimization of free energy one can obtain truly isothermal solutions [9]. In this case, however, the conditions for the existence of stable isothermal equilibrium are found to be much stricter. For an unsoftened $1/r$ potential, an isothermal gravitational system which obeys classical mechanics will contract to a point. For softened systems (with which we shall be concerned here), other isothermal solutions exist. When μ is very large (of the order of the inverse of the softening length), these overlap with the Lane-Embden isothermal spheres. For smaller values of μ the latter only represent *local* entropy maxima. Thus, as shown by Kiessling [9], even systems satisfying Eq. (2) may undergo gravothermal catastrophe. In this case, Lane-Embden ''isothermal spheres'' no longer represent true isothermal equilibria. To distinguish them from the latter, such states will, in general, be referred to as *thermal* equilibria (or isothermal spheres). Other authors have doubted the very validity of the use of entropy maximization or canonical ensemble statistical mechanics (on which all of the above conclusions are based) in systems with long range forces [10]. The validity of relation (2) will be one of the issues we will be examining numerically in this paper.

The existence of a final equilibrium state provides an idealized situation, whereas the dynamics of gravitational systems can be studied for long times by means of particle simulations, without the macroscopic time dependence necessarily characteristic of the evolution of open gravitational systems. This may, for example, be useful in studying the stability of trajectories of particles in N-body systems, their ergodic properties, the diffusion rates of their action variables, and the possible relationship between these properties (for more details concerning the motivations behind such an exercise see the concluding section of Ref. [11]). So far this has been done only for one-dimensional (1D) gravitational systems [12]. It is not clear, however, how the behavior of these systems, consisting of a series of infinite sheets, relates to that of generic three-dimensional gravitational ones. In particular, the force between the ''particles'' in these systems is constant most of the time, but discontinuous when they cross each other; thus much of the body of rigorous results concerning the stability of dynamical systems, which is evidently useful in understanding the properties mentioned above, does not apply [13]. Nevertheless, we will find some interesting parallels between the behavior of these systems and the three-dimensional softened systems described here. Comparison between the behavior of the two types of systems may thus guide further exploration of relevant properties of 1D systems (which may be simulated more accurately for much longer times).

The existence of a well defined relaxation time provides a clean test for the relaxation theory of gravitational systems, and insight into the nature of the relaxation process. This is important for the following reasons. First, there is still much debate as to how a system actually achieves a relaxed *dynamical equilibrium*. Violent relaxation, the original mechanism suggested for this process, is still not very well understood [2,14]. It is still not clear whether a dynamically relaxed final state is necessarily related to this process, or whether the conditions for violent relaxation did exist in the early history of most stellar systems. The second reason is related to the question as to under which conditions (if any) classical ''collisional'' relaxation theory [1,2,15–17] holds. Controversial since its original formulation [18], due to its description of discreteness effects as independent two body encounters added to the mean field motion (something that cannot be *a priori* justified in a nonlinear system), it is nevertheless commonly accepted as valid. Although there seems to be some justification to its use in describing the energy relaxation of particles in N-body gravitational systems [19], and indeed much of the limited numerical evidence [20–23] seems to point in that direction, it seems hardly justified that the validity of this time scale be taken for granted (as is almost always done in research on dynamical astronomy) without strict tests, pending rigorous theoretical justifications for the approximations made in deriving it [24]. Indeed, it is not uncommon to find in numerical simulations that significant effects arising from discreteness noise take place on a scale much shorter than the standard two body relaxation time [25].

Whether it does predict the energy relaxation time correctly or not, the dynamical picture upon which standard relaxation theory is based appears to be flawed, since it implicitly assumes that gravitational N-body systems are integrable and remain so under perturbations due to discreteness. This is modeled as *additive* stochastic noise. The final result is a simple linear superposition of two independent

solutions—the regular motion in the mean field and that under the influence of the stochastic force. However, it is now well known [26–28] that large N-body gravitational systems display sensitivity to changes in their initial conditions which are characteristic of chaotic dynamical systems, and that this appears to be related to their being systems with predominantly negative configuration space curvature [29,30,19,11,31]—thus having qualitative properties very different from those of integrable systems. It is therefore plausible that quantities that depend on the details of the N-body trajectories (as opposed to quantities like energy which is a scalar path independent integral of motion) may relax on time scales that are different from the standard relaxation time [32]. This may affect important observable quantities, like the degree anisotropy in a given system for example. In addition, there are situations when it is clear that the relaxation phenomena involved are beyond the applicability of simple classical two body relaxation theory. Examples of such effects include those arising from the motion of massive particles in an N-body system [2], and the interaction of discreteness noise with the global mean field modes of a system [33]. Theories concerning such effects are far less well established than standard two body relaxation (whereas the approximations made, although not resting on rigorous theoretical grounds, are at least familiar from the theory of stochastic processes and well formulated [34]).

This paper has the basic aim of testing, through direct N-body simulations, some of the theory concerning isothermal spheres, and showing how conclusions of important physical interest—mainly concerning the dynamical relaxation of gravitational systems—can be obtained from this type of study. Thus we show the existence of stable N-body realizations of isothermal spheres for single mass systems, and determine the characteristic time scales for achieving this state (Sec. IV). We also examine the time scale of relaxation towards isotropy for systems with initially anisotropic velocity dispersion tensor (Sec. V). By simulating multimass systems where heavier particles tend to reside toward the center—thus modifying the isothermal sphere density profile —we check if, for a given total energy, mass, and radius, such configurations have a unique density distribution (Sec. VI). In the process, the time scale of mass segregation is evaluated. Finally (Sec. VII), we will check if the condition distinguishing N-body systems that collapse from ones that find a stable thermal equilibrium state is similar to that for the existence of a stable self-gravitating ideal gas sphere [Eq. (2)]. We start by describing the direct summation gravitational N-body code used in this paper, and other technical details like the generation of initial data and the units used (Secs. II and III).

II. NUMERICAL METHOD

The main computational load in the integration of a gravitational N-body system lies in the calculation of the particle-particle forces, which, when summed directly, take a time proportional to the square of the particle number to compute. Many different techniques have been devised in order to speed up the force calculations (for example ''tree'' techniques [35] and adaptive particle mesh methods [36]). As usual, there is always a tradeoff between computational efficiency and accuracy. While the aforementioned techniques are very powerful (the CPU time spent in the force calculations scales as $N \log N$ at the worst), they are not very accurate—one does not expect that, in general, the particle-particle interactions will be calculated to an accuracy much better than few percent. This makes them unsuitable for work in which high accuracy is required, or when the nature of the motion under the influence of gravitational forces is itself the object of study.

The NBODY2 code [37] used to run the simulations described in this paper is one of the many efficient routines devised by Sverre Aarseth, and which are kindly provided by him upon request. It is a direct summation code which uses individual time steps for each particle in the simulation [38], and speeds up the force calculation by using the Ahmad-Cohen [39] neighbor scheme which, in the spirit of tree techniques, separates the force calculations for neighboring particles and those further off—albeit in a somewhat different manner than tree methods. These improvements take into account the very different natural times ($\sim 1/\sqrt{\rho}$, ρ being the local density) in a gravitational system, and the fact that, at a given point, the *irregular* force due to nearby neighbors varies much faster than the *regular* force due to particles further off.

The errors in the calculations (as measured by energy conservation) are controlled by an accuracy parameter η which determines the size of the integration time steps. These errors are constant for values of η below 0.01, and increase as η^2 for higher values [40]. We have found that a value of $\eta_{\rm irr}=0.02$ (controlling the irregular time step) gave reasonably accurate results while maintaining efficient running of the code (both of these aspects depended on the type of enclosure bounding our systems, as we will see below). The tolerance parameter for the regular time step was taken as $\eta_{\rm reg}=0.04$. The softening length was fixed at $\epsilon_p=\frac{1}{500}$.

To perform any experiments on closed systems, we obviously have to find a practical numerical procedure for enclosing them. There is of course more than one way of doing this. For example, one can just reverse the radial component of the velocities of particles that are found to be beyond a certain radius. Alternatively, one can impose ''periodic boundary conditions,'' a situation in which particles escaping from one side of a system reappear on the opposite side. Both these conditions, however, destroy the smoothness of the dynamical system under consideration—in the first case the force can become infinite, while, in the second, some of the variables become discontinuous. It is upon the assumption of smoothness that many of the rigorous results of dynamical systems stability theory and the conclusions drawn from it are based [13]. These may be important in understanding some properties of gravitational dynamics, since it appears that large gravitational systems are close to smooth hyperbolic ones, and it has been suggested that this property may play a role in determining some of their dynamical characteristics [29,30,19,11,31].

In order to conserve the differentiability of (softened) N-body gravitational dynamical systems, we have opted for the following procedure. We surround the system by an ''elastic shell,'' in the sense that a particle venturing beyond a given radius $r=r_0$ experiences a restoring force

$$\mathbf{F}_{\rm res} = -mK(r-r_0)^{2n-1}\hat{\mathbf{r}}, \tag{3}$$

where $\hat{\mathbf{r}}$ is the radial unit vector at the particle's position, and m is its mass. This central force law ensures the conservation of both the energy and angular momentum of individual particles—an important property if one is studying relaxational phenomena.

Although the choice of the constants K and n is to a large extent arbitrary, K has to be chosen so that particles do not venture too far beyond $r=r_0$. Therefore, for large excursions, the force has to be strong. However, if the force rises too steeply at small excursions beyond r_0, large errors in the energies of the particles can result (as these gain or lose energy during their entries into and exits from the $r>r_0$ region when their velocities are largest). After a few trials, values of $K=1300$ and $n=4$ were adopted. This ensured that, for the accuracy parameters adopted (see above), the total energy change over 100 crossing times was always less than 1.5%, and that particles almost never ventured beyond $r-r_0=0.2$ and very rarely beyond $r-r_0=0.1$. The choice also ensured that the inclusion of the boundary force did not slow down the computation too much (the system of units used is described below).

III. INITIAL CONDITIONS AND RELATED PARAMETERS

Throughout this paper we use the units of Heggie and Mathieu [41], whereas the total mass and the gravitational constant are set to unity. Except for some of the runs in Sec. VII, where the initial density decreases according to a power law in the radius, we start our simulations from homogeneous spatial initial conditions with the boundary at $r_0=1$, thus fixing the potential energy at about $\frac{3}{5}$ (give or take effects due to particle noise and softening). The total energy is then determined by the initial virial ratio (T/V). If this is equal to 0.5 (virial equilibrium in the absence of enclosure), then the *mean crossing time* is calculated from [37]

$$T_{cr} = M^{5/2}/(-2E)^{3/2}, \tag{4}$$

which amounts to two time units. If the $T/V>0.5$, the crossing time is shorter. However, considering that the crossing time is only defined as an order of magnitude quantity and that our systems will usually have virial ratios not too different from 0.5, we stick to this definition.

Systems were started from several initial velocity profiles. In the more frequently used distributions, the velocity vectors take random directions, with their magnitudes either decreasing or increasing with radius according to the exponential law

$$V = V_0 e^{-pr}, \tag{5}$$

where the central velocity is fixed by the chosen virial ratio and radial density distribution. The parameter p either takes a value $p=1$, in which case the velocity decreases toward the outside, or $p=-1$, for systems started with a ''temperature inversion,'' so that the velocity at the edge of the system was e times the velocity at the center. We also examined the evolution of systems starting from anisotropic initial conditions with the anisotropy increasing toward the outside according to the prescription

$$V_l = V_{l0} e^{-s\sqrt{l}r}, \tag{6}$$

where $l=1$ and 3 denotes the Cartesian coordinates x, y, and z, respectively. The factor s takes a value of $s=1$ unless otherwise stated. In one of the runs (Sec. V), the system is started with two of the velocity coordinates (for all particles) set to zero.

In some of the runs it was important that our systems did not start too far from virial equilibrium, so that violent relaxation is (presumably) not effective. There is a well known formula (Ref. [1], Eq. 8P-2) for the condition of virial equilibrium of a system on which external forces are applied. Unfortunately, however, when the contribution of the external potential is included in the calculation of the virial ratio, this quantity is highly fluctuating (due to the large contribution from only a few particles which are beyond $r=r_0$). Systems starting from virial equilibrium including the boundary force did not conserve that equilibrium. However, after looking at the long term evolution of the ratio $vir = 2T/V$ in systems started this way, it was found that, for systems with initially homogeneous density, this quantity settled to a value between $vir=1.32$ and $vir=1.38$. In actual trials it was found that isotropic systems started from $vir=1.38$ and from homogeneous density states conserved this quantity to high accuracy (better than 3%) [42]. In the anisotropic case described by Eq. (6), *vir* was conserved to better than 6% during the evolution.

For all systems for which the initial value of $vir=1.38$, the total energy is $E=-0.185$ (this value includes the effect of a softening parameter in the Newtonian potential on the total energy). Therefore, according to relation (2), these systems should evolve toward stable isothermal sphere configurations. vir will take this initial value (corresponding to a virial ratio of 0.69) for all the runs studied here, except for some of those in Sec. VII, where we vary the energy by changing the virial ratio, in an attempt to examine the validity of Eq. (2).

To quantify the departure from thermal equilibrium, we either divide the region where $r \leqslant r_0+0.1$ into ten cells—with the central cell enclosing a radius which is twice the thickness of the surrounding shells—and calculate the trace of the velocity dispersion tensor at time t σ_t in each of these cells. Or, alternatively, we divide the particles into ten groups depending on their distance from the center. Thus the first set would contain the $N/10$ particles closest to the origin, the second set the following $N/10$ particles as one moves outwards, and so on. The second method has the advantage that the sets have equal numbers of particles, while the first procedure has (except for the inner cell) fixed spatial resolution and therefore ensures that there are no large velocity gradients within the individual cells. In both cases we calculate the rms dispersion of the trace of the *velocity* dispersion tensor from the relation

$$\sigma_d = \frac{\sqrt{\frac{1}{10}\sum_{l=1}^{10}(\sigma_t^l - \overline{\sigma_t})^2}}{\overline{\sigma_t}}, \tag{7}$$

where a bar denotes an average over the cells. Usually the value of σ_d using the two different methods of binning the data agreed reasonably well (within 10% and 20%). In what follows when we speak about the ''dispersion'' or about σ_d, we will usually mean the average of σ_d calculated using the two different binning procedures described above.

IV. APPROACH TO THERMAL EQUILIBRIUM

In this section we describe the evolution towards thermal equilibrium of systems consisting of 2500 equal mass particles starting from some of the initial conditions described in Sec. III. Our object is to examine the existence of stable N-body isothermal spheres when simple thermodynamical theory predicts their existence, and to study the time scale for achieving such equilibria when they exist. The choice of the number of particles was a compromise between the divergent requirements of studying systems in which the relaxation times toward dynamical and thermal equilibrium should be significantly different (at least according to standard relaxation theory), and the large computing resources needed by direct summation codes for large N.

A. Case when an intermediate dynamical equilibrium is reached

Figure 1 shows the evolution of σ_d for a system starting with isotropic initial velocities decreasing exponentially with radius [according to Eq. (5)]. One can see that σ_d decreases monotonically (give or take random fluctuations) but only over many crossing times, and reaches a value of 10% or so over a time scale comparable to the two body relaxation time of standard relaxation theory. According to that theory, the half mass relaxation time is given by [15]

$$t_{rh} = \frac{0.14 \times 2500}{\ln(2500\gamma_l)} \times r^{3/2}. \tag{8}$$

After the first few crossing times, during which the system moves away from the spatially homogeneous initial distribution and reaches its slowly evolving dynamical equilibrium state, the half mass radius settles down to a value of about 0.65. Formula (8) therefore predicts a relaxation time scale in the range between about 36 and 45 crossing times, depending on whether one uses the value $\gamma_l = 0.4$ of Farouki and Salpeter [23] or $\gamma_l = 0.11$ of Giersz and Heggie [21]. Looking at the plot in Fig. 1, one can see that during this time interval there is a turnoff in the σ_d time series, so that the decrease in its value with time is much slower. What happens is that, beyond $t = t_{rh}$, the velocity gradient in the central area is, for all practical purposes, smoothed out. The remaining error is due to a slight gradient in the outer regions (which is also eventually smoothed out, but on a longer time scale of about 20 additional crossing times) [43].

Thus it would appear that this system evolves toward the final equilibrium on the thermal time scale, and that this is

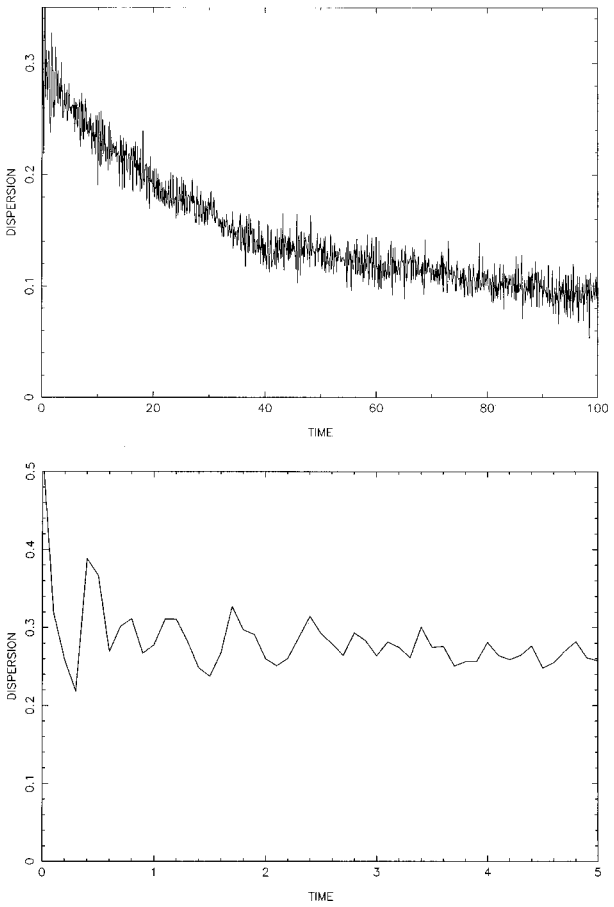

FIG. 1. Time (in units of τ_c) evolution of the relative dispersion of the trace of the velocity dispersion tensor for a system starting from homogeneous density and isotropic velocity distribution decreasing with radius according to Eq. (5) with $p = 1$.

adequately described by standard two body relaxation theory. As mentioned in Sec. I, this theory may be adequate in describing energy relaxation (which is presumably the main mechanism acting here), even though the dynamical picture it is based on may be rendered inaccurate by the chaotic nature of N-body trajectories. Nevertheless, it has been suggested [29] that the exponential divergence of phase space trajectories accompanying such motion *itself* induces a relaxation time scale—which could be different from the standard two body time since it is obtained from entirely different considerations—by estimating the time scale for smoothing out (or otherwise radically modifying) the phase space density distribution (''mixing''), as suggested by the ergodic interpretation of statistical mechanics [44,45]. However, the exponentiation time scale in gravitational N-body systems is found to be of the order of a dynamical time [27,28,11], so if it did directly correspond to the evolution toward the thermal equilibrium state, one would expect such a state to be reached within an accuracy of a few percent after a few crossing times (See Eq. (29) in Ref. [19]). The fact that this is obviously not the case has led some investigators to conclude that the exponential divergence of N-body trajectories has nothing to do with their relaxation [27], or that it is more likely to *only* affect the time scale for achieving *dynamical* equilibrium [46]. For the system just described, this corresponds to the time scale for reaching the slowly

evolving state which is achieved when σ_d has gone down from an initial value of 0.47 to a value to about 0.3. This happens during the first few crossing times (Fig. 1).

The above conundrum may apparently be resolved by noting that, for large-N systems in dynamical equilibrium, the (mixing) time scale for obtaining an equilibrium phase space density distribution is not simply a few exponentiation time scales, but may instead be given by [11]

$$\tau_r = 2\sqrt{3}\ln(1/d)T\sqrt{N}\tau_e, \tag{9}$$

where d is the linear phase space resolution over which averaged (coarse grained) distribution functions should not evolve after a time of order τ_r. We calculated the exponentiation time scale τ_e for the systems studied here by using the Ricci curvature method [48,19,11]. It was found to be about 0.4 crossing times. The kinetic energy is also $T\sim 0.4$. Thus, for a resolution of, say, 10%, the time scale given by Eq. (9) is ~ 67 crossing times, which is compatible with the results described here. Thus it will be necessary to examine the variation of τ_r with (sufficiently large) N to see if the above formula has any validity in determining the evolution toward equilibrium, or whether the standard theory holds. The results of such simulations are currently being analyzed [49].

B. Case when the local dynamical equilibrium is close to the thermal equilibrium state

The situation was found to be very different for the system starting with a temperature inversion [i.e., with $p=-1$ in Eq. (5)]. In this case, the long lived steady state arrived at on a time scale comparable to the exponentiation time scale *is* the thermal equilibrium state. As can be seen from Fig. 2, this state is reached in essentially less than a crossing time. After that, σ_d deviates from zero by only a few percent. This simply corresponds to the particle noise. It is to be stressed that here, as in the case when the initial velocities decreased with the radius, the virial ratio remained nearly constant throughout the evolution. The density distribution and parameters that sensitively depend on it (like the potential energy) changed only slowly, with a rate comparable to that of the slowly evolving system described in Sec. IV A. We conclude, therefore, that this equilibrium state was not reached through the conditions normally associated with violent relaxation [2]. It also could not have been reached through standard two body relaxation, since this process takes place on the much longer time scale of Fig. 1.

It has been argued [11] that the exponential divergence of trajectories of gravitational systems in dynamical equilibrium mainly serves to smooth out the $(6N)$ phase space density distribution on subspaces compatible with the dynamical equilibrium, and causes only slow diffusion away from that state. Thus while Eq. (9) gives the relaxation time scale for a system where the phase space divergence of trajectories is such as to maintain a given dynamical equilibrium—i.e., by covering different configurations of a slowly evolving equilibrium—if such an equilibrium is not found, the divergence can take place at arbitrary directions in the phase space. Because of the exponential divergence normal to the phase space motion, a dynamical equilibrium with smooth macroscopic density and velocity distribution may be quickly attained even if the conditions for violent relaxation (which require large density fluctuations and far from equilibrium evolution) are not satisfied. Once a dynamical equilibrium is found, the geometry of the configuration space must behave in such a way that *globally* the exponential divergence is effectively *along* the motion (although locally it must be normal). Only the small scale fluctuations away from the equilibrium due to the discreteness noise modify this. However, evolutionary time scale related to these is much larger [50].

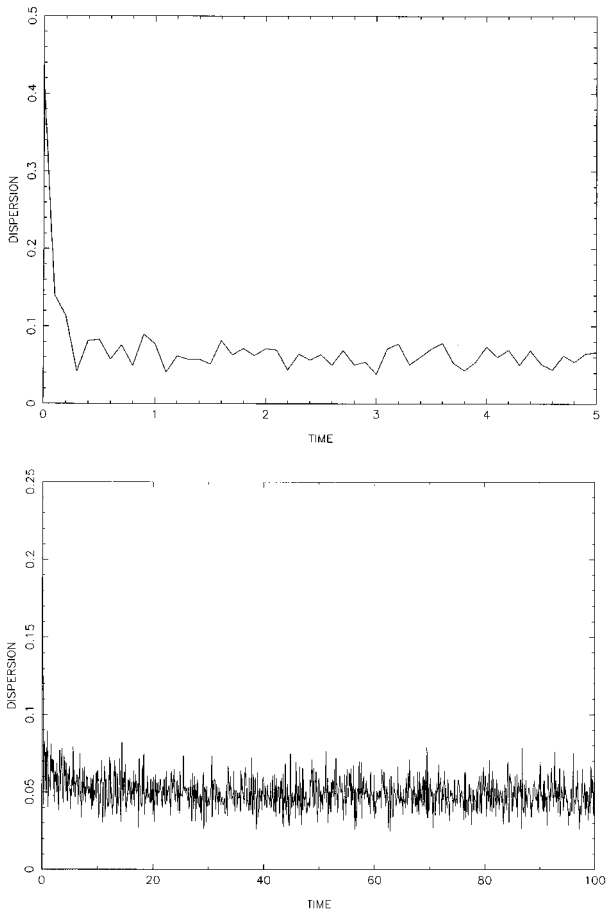

FIG. 2. Time (in units of τ_c) evolution of the relative dispersion of the trace of the velocity dispersion tensor for a system starting from homogeneous density, and an isotropic velocity distribution increasing with radius according to Eq. (5) with $p=-1$.

It is worth noting here that behavior similar to what has been observed here occurs for one-dimensional gravitational systems. In this case, too, systems prepared near stable dynamic equilibria gradually approach the thermal equilibrium distribution due to collisional relaxation [12,51], while those initialized from other configurations rapidly relax to a state very close to thermal equilibrium [52].

C. Invariance of the final thermal equilibrium state

The final equilibrium state should, by definition, be invariant under the effect of gravitational interactions between particles, and, therefore, if it is thermodynamically stable, one does not expect it to evolve [47]. As can be seen from Fig. 2, this is actually the case. After the initial relaxation phase, during which σ_d drops from its initial value (of about

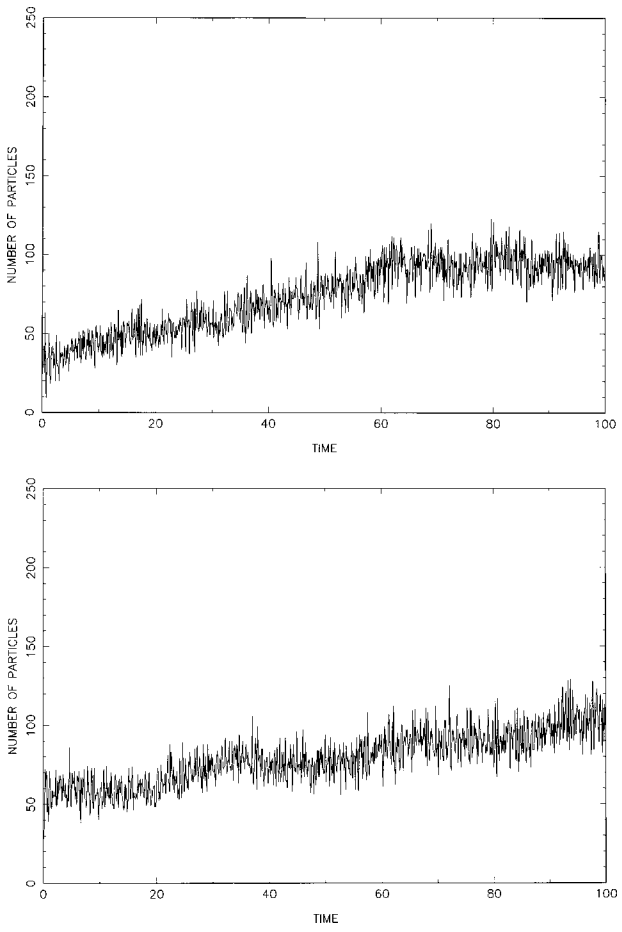

FIG. 3. The evolution of the number of particles in the central cell ($r=0.2\times r_0$) as a function of the (crossing) time, in runs corresponding to the plots in Figs. 1 (top) and 2.

0.32) to an average value of about 5% (consistent with fluctuations arising from the particle noise), the value of σ_d hardly evolves at all.

Further evidence that this final state is long lived, and that no gravothermal catastrophe occurs, can be obtained by looking at the evolution of the particle number in the central spatial cells of the two systems we considered. This is shown in Fig. 3, after an initial increase as the system settles toward the isothermal configuration (starting from the homogeneous density distribution), the particle number in the central cell evolves only slowly, and then only in a limited way which does not signal any gravothermal collapse, but instead an evolution toward a more detailed equilibrium. Thus, while evolution toward the equilibrium distribution in velocity space is fast, evolution in configuration space is relatively slow. This again illustrates the multiple time scales present in gravitational systems. For the system starting from the initial state where the velocities decrease outwards, the central density ceases to evolve significantly after about 60 crossing times.

For every set of numbers M, E, and R, there exist a unique stable Lane-Embden isothermal sphere configuration (cf. Fig. 8-1 of Ref. [1]). In our runs, $M=1$ and $E=-0.185$. Since very few particles (usually not more than ten) are present beyond $r-r_0=0.2$, we take this to be the boundary value $R=1.2$. In this case, the central density of this unique isothermal sphere is given by [1]

$$\rho_0=\frac{3\sigma}{4\pi G r_c^2}, \tag{10}$$

where, in the units used in this paper, $G=1$. For the above value of the total energy and $vir=1.4$, the velocity dispersion $\sigma=0.186$. Here r_c is the core radius of the system (within which the density is roughly constant). For the above values of the parameters $r_c\sim 0.3R$ (this, in fact, can be inferred from Table 4-1 and Fig. 8-1 of Ref. [1]). One can then deduce that inside $r\leqslant 0.2$ there will exist about a hundred particles of mass 1/2500. This is indeed approximately the number of particles found in the central cell (that is, for $r\leqslant 0.2$) at the end of our simulations, which suggests that the resulting end states are indeed thermal equilibria.

D. Approach to equilibrium in anisotropic systems

To conclude this section, we finally note that the evolution of σ_d for the system starting with anisotropic velocities [prescribed according to Eq. (6)] turns out to be intermediate between the two cases described above. The dispersion decreases within a couple of dynamical times to a value of about 12%, after that the system continues to evolve to a more precise thermal equilibrium state but in a less rapid manner, taking about 65 crossing times for the dispersion to hover around an average $\sigma_d\sim 5\%$. The initial anisotropy is, however, washed away within a time scale of 10–20 crossing times. This a prelude of things to come (Sec. V), when it will be confirmed that the relaxation of velocity anisotropies appears to be much faster than that of the energies.

V. RELAXATION OF VELOCITY ANISOTROPIES

Energy relaxation is one of the effects produced by discreteness noise in N-body systems, but there are others. An important one is the relaxation of initially anisotropic velocities. This, in a spherical system, will correspond to angular momentum relaxation. As mentioned in Sec. I, it is not evident that the energy relaxation time scale, which should correspond to the long relaxation time scale of Sec. IV, is the only time scale of interest when examining effects induced by the discreteness of N-body systems. It is important, therefore, to check how other quantities, which may be more directly related to the detailed particle trajectories, tend toward equilibrium values.

It was seen near the end of Sec. IV that the relaxation of velocity anisotropies can be much faster than the thermal (energy relaxation) time scale when the anisotropy is initially mild. In this section we study the evolution of velocity dispersions for systems where these are initially strongly anisotropic. To do this, we simply start our system with the y and z velocities set to zero. The x velocities, which do not vary with radius, are scaled in such a way as to keep the same virial ratio as before. The system starts from a homogeneous density and will therefore have the same energy as the ones discussed in previous sections. This configuration is obviously very far from dynamical equilibrium and our system ''violently relaxes,'' with the virial ratio oscillating wildly for many dynamical times. It settles to a value of about

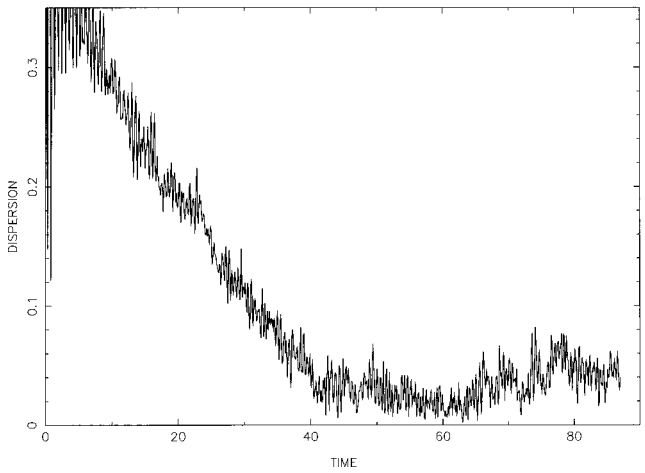

FIG. 4. Relaxation of initial velocity anisotropies. The system is started with zero y and z velocities, and achieves a quasisteady state after $\sim 10\tau_c$. Time is measured in units of the crossing time τ_c.

0.73 ($vir = 1.46$) after roughly ten crossing times; this will therefore be our zero point in calculating the relaxation time.

After the ten initial crossing times, the trace of the velocity dispersion tensor does not vary significantly with radius ($\sigma_d \sim 6\%$ almost independent of r), and in this sense the system is isothermal. Moreover, this is mostly true in each direction x, y, and z *independently*. We can therefore average our anisotropy measure over all radii to obtain less noisy data. For easy comparison with energy relaxation, for this purpose we will use the same formula [Eq. (7)] as before, replacing the trace of the dispersion tensor in each radial cell (σ_t^l) with the sum of the velocity dispersions over all cells in the x, y, and z directions. The number 10 is then replaced by 3, and $\overline{\sigma}$ is taken as the average of σ_x, σ_y, and σ_z.

The results are shown in Fig. 4. After the initial ''violent evolution'' of the first ten crossing times, following which the dispersion settles to a value of ~ 0.32, it starts decreasing abruptly (almost linearly), reaching the equilibrium (isotropic) value after a total of ~ 40 crossing times. This means that the *effective* relaxation time (after achieving dynamical equilibrium) toward isotropic velocities is about 30 crossing times. This is significantly smaller than the time required for complete energy relaxation (about a quarter of that time). Thus it would appear that, at least for the case of closed systems (and for the initial conditions examined here and in Sec. IV), relaxation of initial anisotropies seem to be *much faster* than energy relaxation.

However, three obvious and related questions will have to be addressed before the above conclusion is credible. (i) Does the system become triaxial during its evolution, and, if so, can much of the relaxation be due an abundance of chaotic orbits in the mean field? (ii) If the system is not spherical during its evolution, what is the effect of the spherical enclosure on the relaxation of such a system? (iii) Are the density distribution in the system studied in this section, and the ones for which energy relaxation was examined, similar (this is important, since the relaxation is expected to depend on the local density)?

To answer the first question, we have calculated the potential energy tensor [1] during the evolution. From this, we infer the axis ratios (assuming an ellipsoidal density distribution with constant axis ratio). The answer is yes; as may be expected, a system starting from such initial conditions is not spherical during its early evolution. However, it is only mildly triaxial at ten crossing times (with the two longest density axes having ratios $a/b = 0.88$, which means still larger equipotential axis ratios), and is almost completely axisymmetric ($a/b = 0.96$) after 20 crossing times. Since such mild triaxialities are not likely to cause a rapid evolution (without strong central mass concentration [54]), and since the evolution does not slow down after the asymmetry is almost completely lost (at ~ 30 crossing times), and the departure from sphericity is very small, we conclude that such mean field evolution would not appear to play a central part in the relaxation process.

One may suspect that the presence of an artificial spherical box surrounding an asymmetric system may play a role in its evolution toward a spherical shape, which may be accompanied by velocity relaxation toward isotropy. However, this does not appear to be the case here: since the whole process of evolution toward isotropy takes place in a fraction of the energy relaxation time, one expects most particles to conserve their energy and be confined away from the spherical enclosure (by their zero velocity surfaces). In this case, particles in the outer areas would be the most affected, and would then ''communicate'' the disturbance felt to the inner areas. It follows that one would expect the evolution toward isotropy to take place from outside in—i.e., the outer regions become more isotropic before the inner ones. However, this was found not to be the case; if anything, it was the inner areas which evolved slightly faster (presumably due to their higher densities).

In principle, by virtue of the uniqueness of isothermal spheres for a given value of the parameter μ, the density distribution of all final equilibrium configurations with the same total energy, radius, and mass should be the same. However, we have allowed for our boundary to be elastic (thus R is not completely fixed), and the energy in our N-body simulations is not exactly conserved. Therefore, the density profiles may vary slightly from one model to another. In particular, one expects the system studied in this section to be more centrally concentrated due to its initial violent relaxation [55]. This indeed turns out to be the case: the present system is more centrally concentrated. However, it is only significantly so in the innermost cell or two. Beyond the fifth cell the reverse is actually true. We can again appeal to the uniformity of the relaxation toward isotropy at all radii (including the mid regions where the density in both systems is similar) to suggest that the increased central density is not an important effect.

In addition, all the above problems do not occur for a system with mild initial velocity anisotropies and gradients (discussed at the end of Sec. IV). This configuration stays almost spherical during the evolution, thus the relaxation toward isotropy means a relaxation in angular momentum (which is conserved by our boundary force law). This takes place *concurrently* with the relaxation of a thermal gradient (so that there is no question of different density distributions affecting the relaxation rate). Thus it appears that the effect of relaxation toward isotropy being faster than energy relaxation is real. However, it is still left to future, more precise, simulations with more controlled conditions, and exploration

of the parameter spaces and variation of particle numbers, to confirm this potentially important result.

The idea that relaxation of velocities toward a Maxwellian (which is what happened here when an isotropic state was reached) may be faster than the process of energy relaxation is actually quite old [56,57]. In their study, Prigogine and Severne used a kinetic formulation that avoided the artificial cutoff introduced to eliminate the divergence in the Coulomb logarithm. They found that relaxation toward a Maxwellian is $\sim \ln N$ times faster than energy relaxation. This is compatible with what is found here. Moreover, their analysis also suggested the oscillations around the equilibrium shown by the longer time evolution of the time series in Fig. 4.

VI. EVOLUTION OF MULTIMASS SYSTEMS

The main difference between a single mass system and one consisting of particles with different masses is that, in the latter case, equipartition of energy will cause heavier mass particles to spiral toward the center of the system and reside there. General thermodynamical considerations show that once the total mass, energy, and radius of an isothermal sphere are fixed, there corresponds only one configuration which is stable, and this has a unique value for a given ratio of the density at the center and at the boundary (cf. Ref. [1], Fig. 8-1). One therefore expects that a perturbation which conserves μ [cf. Eq. (2)], but changes the ratio of central to boundary densities, will necessarily cause a system to move away from this unique thermal equilibrium configuration. If the system cannot move back (by restoring the initial density ratio), the equilibrium is unstable. This is of course precisely the situation here, where there is a mechanism that acts so as to cause high mass particles to spiral to the center, leaving μ unchanged. Since this process is due to equipartition of energy between particles, it is statistically irreversible. Gurzadyan, Kocharyan, and Matinyan [58] examined the evolution of the transition from the case when a thermal equilibrium existed and the case when it did not exist. The bifurcation between these cases was found to depend on the existence of a central mass. In particular, it was found that as one increased the central (point) mass, the value of μ required to achieve a stable isothermal thermodynamic equilibrium became significantly larger. That is, the aforementioned curve in Ref. [1] shifted upwards. This means that, other parameters staying constant, one has to continually increase the system's energy in order to maintain thermal equilibrium as the central density is increased.

Thus one expects the increase in the central density with time during the evolution of isolated multimass systems to lead these in the direction of gravothermal collapse. Figure 5 shows σ_d for models with velocities scaled according to Eq. (5), and where the mass distributions follow a Salpeter mass function [59] with the highest mass particle being ten times more massive than the lightest particle. As can be seen, neither of the models tends toward a thermal equilibrium.

It is found that, by the end of the run, the average mass per particle inside the central cell is about twice the original value (corresponding to a sample with a random mass distribution). Also, due to the core contraction, not only is the average mass per particle in the central region increasing with time, but also the total number of particles of any mass

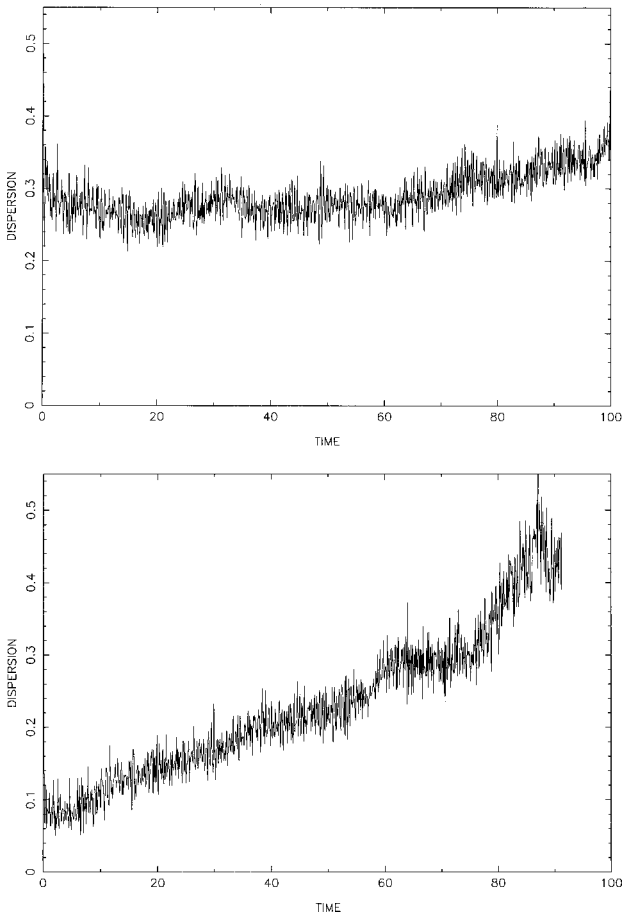

FIG. 5. Evolution of the relative dispersion of the trace of the kinetic energy tensor of the multimass systems of corresponding to Figs. 1 (top) and 2 as a function of the crossing time. Top: system started with velocities decreasing with radius. Bottom: system started with temperature inversion.

(compare Figs. 6 and 3). This process was found to be much faster in the system with initial temperature inversion—which tended fastest toward equilibrium in the equal mass case. In fact, for that system, the number of central particles increases to the point where a large number of tight binaries form, changing the effective specific heat of the system and causing the central density to eventually decrease (the core re-expands). The tightening of these binaries causes numerical difficulties and the simulation had to be stopped (since the NBODY2 code is not designed to deal with such situations [37]). The main reason the evolution is faster for the case when thermal equilibrium is rapidly reached is probably because this equilibrium is more centrally condensed than the intermediate dynamical equilibrium states the thermally evolving system passes through. The higher density means that the accumulation of heavy mass particles happens at a faster rate, which in turn increases the density and velocity gradients, and so on.

It is important to note here that, although the concentration of the heavier mass particles toward the center is due to those losing kinetic energy in gravitational encounters with lower mass ones, true energy equipartition is never achieved. This is because a stationary thermal equilibrium is never reached. Instead, because the system suffers a gravothermal

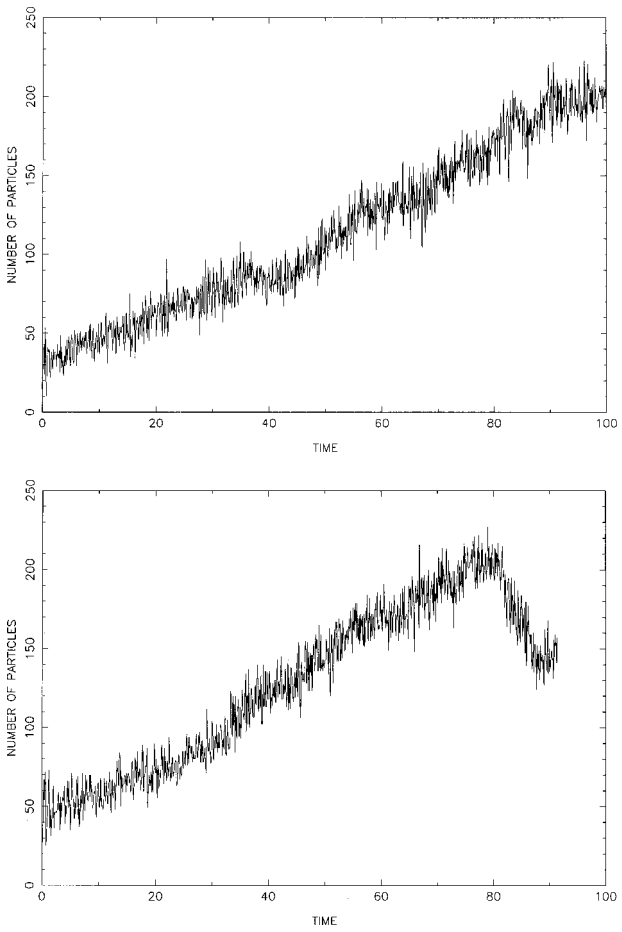

FIG. 6. Evolution of the number of particles in the central cell of the multimass systems of Fig. 5 in terms of the crossing time. Top: system started with velocities decreasing with radius. Bottom: system started with temperature inversion.

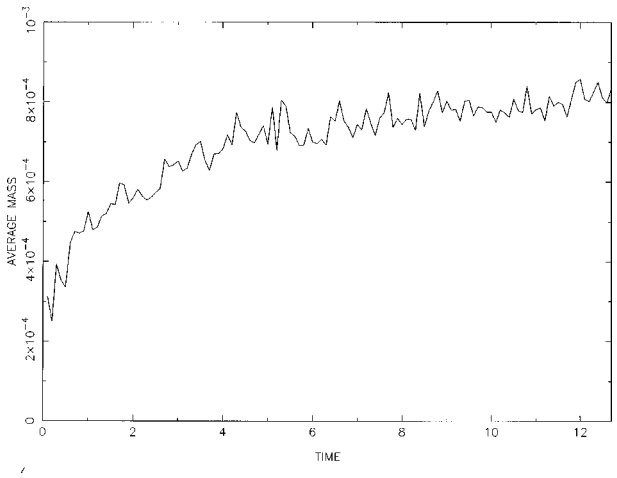

FIG. 7. Average mass per particle inside $0.1r_0$ for a multimass system where the virial ratio varies significantly during the evolution. The total mass of the systems is unity. Time is measured in units of the crossing time τ_c.

catastrophe, it continually evolves toward more inhomogeneous distributions (in velocity and configuration spaces), with the kinetic energy of particles continually varying. The total kinetic energy increases as the core contracts and the virial ratio increases; if anything, the variations in the kinetic energies of the particles are enhanced. This is in contrast to the one-dimensional case (which does not display unstable thermal properties), where equipartition can be achieved [53].

The above results suggest that there are, in principle, statistically irreversible effects that can change the density distribution of a given system away from that of the stable isothermal solution compatible with its radius, total energy, and mass. This in turn suggests that condition (2) is only a necessary condition for a system to evolve to thermal equilibrium. It has been suggested [9] that the aforementioned condition guarantees only that an isothermal sphere is a *local* free energy minimum. It then simply happens that these more concentrated states are dynamically inaccessible (or at least have a low probability of being reached) from certain initial configurations. This is obviously the case for the equal mass systems discussed in Sec. V. However, there could be different configurations of equal mass systems which do not evolve toward the thermal equilibrium state, even when it is predicted to exist. This assertion will be tested and discussed further in Sec. VII.

It was observed that the process of increasing the average mass per particle and the total mass in the central area is effective over a time scale much smaller than the standard two body relaxation time. This was especially so for systems starting from initial conditions that, although in virial equilibrium, are far from dynamical equilibrium, so that a significant change in the virial ratio occurs during the subsequent evolution. We ran a simulation, for example, in which the initial velocity distribution was given by Eq. (6) (i.e., anisotropic velocities with asymmetry increasing toward the outside) but with the exponential factor $s=4$. During the first two crossing times or so, $vir=2T/V$, which starts from the usual value of 1.38, fluctuates between values of 1.24 and 1.68. During this time the particle number in the central cell ($r<0.2r_0$) rises from an initial value of 27 to about 400. After that, *vir* settles down to a value of about 1.54, which changes relatively slowly (to about 1.6 after ~ 13 crossing times), and the number of central particles settles down to about 250. Figure 7 shows the average particle mass inside the sphere with radius equal to half of that of the central spatial cell ($r<0.1r_0$) which is clearly seen to increase over a few crossing times. The mass per particle was found to be above average in all the inner cells and below average beyond the fifth cell. In these outer cells the average mass is ~ 0.35 after a few dynamical times. We stress here that this was found to be true even for the very outer cells, where the the standard relaxation rate is relatively long.

One possible explanation for this remarkably fast relaxation rate of high mass objects could be that the ''test particle'' approach used to calculate two body relaxation time scales fails for sufficiently massive particles. This is because a heavy mass particle moving in a system may induce collective effects, altering the density around it and changing the interaction strength with neighboring particles (see Secs. 13 and 14 in part II of Ref. [2]). This effect appears to be amplified in systems evolving from dynamical equilibrium. It may be worth noting here that this type of anomalously

TABLE I. Values of the relative dispersion in the trace of the velocity dispersion tensor averaged over the last ten crossing times for 500 particle systems. Runs normally lasted for a total of 100 crossing times, except for the run with with $E=-0.491$ which was terminated at 33.9 crossing times due to numerical difficulties

T/V_i	E	$\overline{\sigma_d}$
0.69	-0.185	0.11
0.60	-0.238	0.11
0.50	-0.298	0.12
0.40	-0.357	0.16
0.30	-0.417	0.22
0.50	-0.491	0.50

rapid increase of high mass particles in violently relaxing objects has previously been reported [60,22].

VII. PHASE TRANSITION OF ENCLOSED EQUAL MASS *N*-BODY SYSTEMS

We would now like to know whether the value of the parameter μ, at which the phase transition between systems that end up as isothermal spheres and those that suffer a gravothermal catastrophe takes place, is close to that given by Eq. (2), and if this will depend on the initial conditions. In the units we employ, $G=M=1$, and the boundary radius is also approximately equal to unity, so that varying μ basically amounts to changing the total energy. If we stick to homogeneous initial spatial configurations, then this will amount to varying the kinetic energy. We choose this to obtain virial ratios of 0.69, 0.6 0.5, 0.4, and 0.3. In addition, one run was started in virial equilibrium ($T/V_i=0.5$) but with the initial density varying as $1/r^2$ instead of being homogeneous (this accordingly decreases the energy).

We compute the average of σ_d over the last 100 outputs (these are produced during the last ten crossing times out of a total of usually about 100). Since here we are interested in the final state of the system and not the time it takes to get there, the number of particles used in the simulations is not very crucial. In fact, using fewer particles ensures that this state is arrived at in a lower number of crossing times. Thus one can be more confident that a final state has been reached in 100 crossing times (say), and use the CPU time saved to be able to examine a larger number of initial states. The velocities were started with a temperature inversion, so that rapid evolution toward a thermal equilibrium state is expected when such a state exists (Sec. IV).

The results, which are shown in Table I, suggest that the transition between systems that tend toward thermal equilibria, and ones that do not, does indeed take place at a value of $\mu\sim-0.33$, in accordance with the results of Antonov [5] and Lynden-Bell and Wood [4]. Thus these studies, which employed simple thermodynamical considerations applied to ideal gas spheres (and were confirmed by local stability analysis in a statistical mechanical context [7,8]), provide accurate results concerning the stability of *N*-body gravitational systems starting from the initial conditions described above.

However, homogeneous spatial initial conditions are obviously, in a sense, closer to the density distributions of isothermal spheres than to collapsed objects (in the sense that these will have to pass through the less concentrated former state before reaching the latter). It was argued [9] on the basis of global minimization of the free energy, that, for real gravitational systems, relation (2) only guarantees *local* entropy maxima, global maxima being collapsed states with large temperature gradients. We have therefore examined the approach to equilibrium from spatial initial conditions where the density decreased according to a power law that is steeper than the inverse square relation associated with an isothermal sphere. It was found that for densities decreasing as steeply as $\sim R^{-2.7}$, the thermal equilibrium final state was recovered. For power laws steeper than that, however, the central concentration increased and the integration stopped (due to numerical difficulties associated with the fact that the NBODY2 code is not designed to deal with such situations [37]). The details (e.g., whether the integration stopped after a few crossing times or went on for a while) depended on the softening, and on whether the initial velocities increased or decreased with the radius. However, even with very large softening $\sim 0.1 r_0$ and an initial temperature inversion, a system starting with density decreasing as R^{-3} could not be integrated beyond $84\tau_c$. It is interesting to note here that the initial velocity distribution (with its strong temperature inversion) was essentially conserved for all these tens of crossing times.

It may perhaps be necessary in the future to repeat these simulations with an *N*-body routine that is better suited for integration of the more centrally concentrated structures of collapsed gravitational objects (e.g., routines involving two body regularization) to be completely certain of the following conclusion. The current evidence, however, suggests that condition (2) indeed only guarantees that thermal equilibrium states are local entropy maxima. These correspond to Lane-Embden isothermal spheres which are not true isothermal equilibria (cf. Sec. I). Systems starting from certain initial conditions may still end up in collapsed states. The condition given in Ref. [9] for this to never happen (i.e., for thermal equilibrium states to be global entropy maxima) is much stricter (with the temperature in units of $k_B=1$, μ must be of the order of the inverse of the softening parameter, i.e., $\sim+500$, to prevent collapse). The truly isothermal solutions overlap with the Lane-Embden isothermal solutions in this range of μ.

VIII. CONCLUSION

In this paper we have studied some aspects of the dynamics of closed gravitational systems by means of direct *N*-body simulations. Stable *N*-body isothermal spheres are convenient laboratories where long term (e.g., ergodic) properties of *N*-body trajectories can be studied without the distraction caused by the eventual evolution of global properties—which is inevitable in open systems. Up to now this type of study has been conducted only in the case of one-dimensional systems. The existence of a final equilibrium state also means that the relaxation time is well defined. Such systems are therefore ideal for testing gravitational relaxation theory. This paper presented an exploratory study where some questions concerning the circumstances under which stable *N*-body gravitating isothermal spheres exist,

and the relaxation times for evolution toward or away from such states were discussed. The main findings are as follows.

(1) It is confirmed that closed equal mass N-body systems that satisfied relation (2) could indeed evolve toward well defined thermal equilibrium states, as expected from simple dynamical and thermodynamic considerations applied to systems with a perfect gas equation of state and from local stability theory [1,2,6,7]. However, it was found that systems starting from centrally concentrated initial states (e.g., with density decreasing as the inverse cubed of the radius) did not evolve toward thermal equilibrium even when the aforementioned relation was satisfied. This effect was predicted by Kiessling [9], who, based on a statistical mechanical analysis which made used of the minimization of the free energy, suggested that a much more stringent relation than Eq. (2) would have to be satisfied in practice for all states to tend toward isothermal sphere configurations. Unless μ is very large (of the order of the inverse of the softening parameter), isothermal spheres are only *local* entropy maxima and do not truly represent isothermal solutions (Sec. I). Whether a particular system will evolve as to end up as an isothermal sphere will depend on its initial conditions. In particular, local stability of isothermal sphere configurations will guarantee that some initial states will tend toward such equilibria when they exist—for example, states that are initially less centrally concentrated than the thermal equilibrium configuration.

(2) The rate at which evolution toward thermal equilibria proceeds depends a lot on the initial conditions. If the system starts sufficiently near a *dynamical* equilibrium state which happens to be different from the true thermal equilibrium, it will quickly (within a crossing time or so) tend toward the dynamical equilibrium. The subsequent thermal evolution appears to proceed on a time scale compatible with slow two-body relaxation. If, on the other hand, a system starts from a state from where the nearest dynamical equilibrium and the true thermal equilibrium are close (this was found to be the case, for example, for systems which started from spatially homogeneous states and with initial temperature inversion), it will tend toward this common equilibrium in a crossing time or so. This can happen even if the conditions for violent relaxation are not likely to be satisfied—for example, when the system under consideration stays very close to virial equilibrium throughout the evolution and is not clumpy [2].

(3) The relaxation of anisotropic velocity dispersions was found to be 3–4 times faster than energy relaxation (which presumably determines the evolution toward thermal equilibrium in the case when this evolution is slow). One therefore concludes that in gravitational systems, different quantities may have different relaxation time scales. In particular, quantities that depend on the details of a given particle's trajectory are likely to evolve at a different rate than its energy, which is a scalar integral of the motion [19].

(4) We have also conducted simulations of multimass systems—other parameters being the same as in the corresponding single mass runs. These systems did not tend to stable thermal equilibria, but instead underwent gravothermal catastrophe (collapse). This appears to be triggered by the statistically irreversible accumulation of heavy mass particles in the central areas, which changes the ratio of central to boundary densities. Since, according to simple dynamical and thermodynamic arguments, given the values of the total mass, energy, and radius, there exists a unique thermodynamically stable isothermal sphere for a given value of the aforementioned density ratio, no such equilibria will be possible when central mass is irreversibly added while keeping other parameters fixed.

(5) The rate at which the average mass per particle increased in the inner 10–20 % of the boundary radius was remarkably fast. In particular, it was found that in a situation when there was significant departure from virial equilibrium (i.e., when violent relaxation may be at work), mass segregation is observed on surprisingly short time scales of a few crossing times (for systems of 2500 particles). Such effects have also been previously observed [22,60].

It is hoped that this paper has demonstrated some uses of numerical experimentation with closed N-body systems, and has illustrated that the study of relaxation in gravitational systems is not itself a closed subject, but is instead a rich and largely unexplored area with many potentially interesting phenomena yet to be revealed. On the theoretical side, an interesting finding is that stable long lived thermal equilibrium states can exist which are *not* entropy maxima. This means that the true entropy maxima (centrally concentrated objects with large velocity gradients), although compatible with macroscopic constraints such as total energy and momentum conservation, are not reached—at least not over time scales comparable to the thermal relaxation times. This of course brings into question the ergodicity of N-body gravitational systems (over the total energy-momentum subspaces) and the applicability of the standard postulates of statistical mechanics to these systems, along with their standard dynamical interpretation.

Two applications to come out of this effort that are particularly relevant to the study of stellar dynamics are the evaluation of the energy relaxation time in cases where a definite thermal equilibrium existed, and the calculation of the relaxation time of initially anisotropic velocity distributions. In the first case, the relaxation time was found to be compatible with that of classical two body theory. However, it was also compatible with a recent estimate [11], derived from the exponential divergence in N-body systems in dynamical equilibrium, in which the relaxation time scales as $\sim\sqrt{N}$. It is then important to check how the relaxation time toward thermal equilibrium states (which provides the only situation whereas the relaxation time in a three-dimensional gravitational N-body system is well defined) scales with N. Results from a study of this type are currently being analyzed [49].

In the second case, we found that relaxation toward isotropic velocity distribution is considerably faster than that of energy relaxation. If confirmed, this would be an important result, with astrophysical applications to the study of such phenomena as the ellipticity distributions of globular clusters. Prigogine and Severne [56], who predicted this effect on the basis of a kinetic formulation of the problem which avoided the introduction of a long range cutoff used to eliminate divergence in the Coulomb logarithm, estimated that the variation with N of the rate of randomization ''of the kinetic energy to that of the transformation [of the] potential energy

into kinetic energy'' (thermal time scale of energy relaxation) goes as $\sim \ln N$. However Prigogine and Severne still retained some of simplifying features reminiscent of the original Chandrasekhar formulation of the relaxation time: infinite medium (i.e., no mean field), weak coupling approximation, two body encounters, etc. Obviously, then, a direct estimate of the variation with N for the aforementioned ratio is also useful.

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